The Air Force Research Laboratory (AFRL)

A Beautiful Game

0	2	-1
2	1	-2
-1	-2	2

Wright-Patterson
Educational Outreach

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Wright-Patterson Air Force Base
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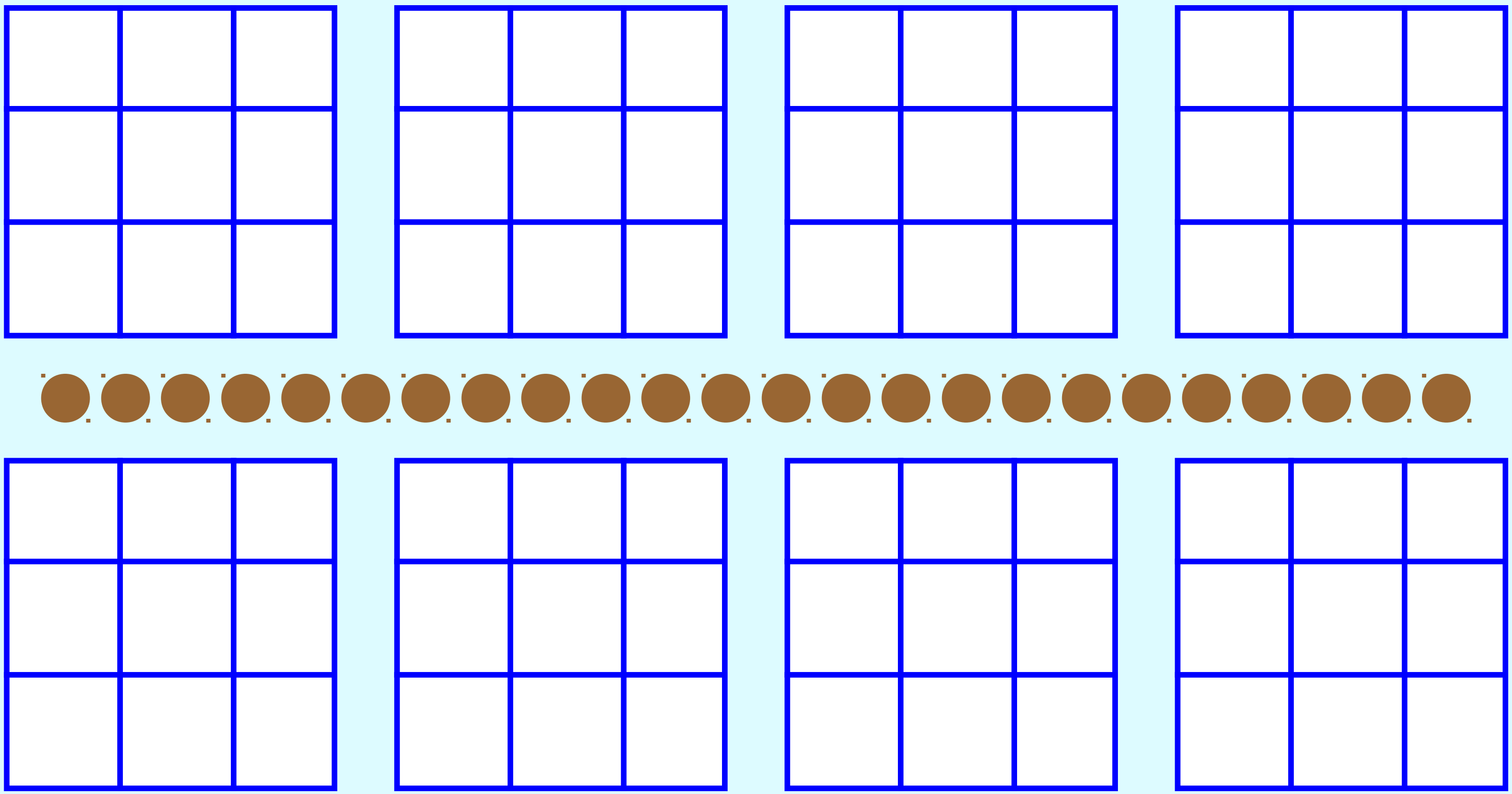
How Does Tic-Tac-Toe Differ From Pitching Pennies?



“Stone-Scissors-Paper”:
An Old Children’s Game

Rules of Engagement

- Stone breaks scissors
- Scissors cuts paper
- Paper covers stone

Player 2
Free-Will Choice

Player 1
Free-Will Choice

	S	C	P
S	0	1	-1
C	-1	0	1
P	1	-1	0

The above is called a game matrix. Payoffs are shown in the body of the matrix. Positive quantities are advantageous to the row player (Player 1) and negative quantities are advantageous to the column player (Player 2).

Some "Game-Theory" Definitions

- Two-Person Game: A game where only two people are playing against each other
  - This is the only type of game that we will study in this class
- Game Matrix: A matrix representation of all possible moves by either player and the associated payoffs
- Zero-Sum Game: A game where all the loss incurred by one player becomes the gain of the second player and visa-versa
- Most Conservative Strategy: An approach to game theory where players strive to minimize their losses
- Optimal Strategy: A strategy, when employed, that guarantees minimal losses
- Value of a Game: The average payoff (loss or gain) when the optimal strategy is employed
- Fair Game: A game where the payoff is zero

Most Conservative Strategy:
Leading to Strictly Determined Game

Player 2: 4 Choices

Player 1: 4 Choices

-4	-2	1	2
-5	-2	-7	1
3	-1	0	3
1	-3	-1	5

Player 1 will want to stick to Row 3 in order to minimize loses.
Player 2 will want to stick to Column 2 in order to minimize loses.

Since the most conservative strategy reduces to a single best move for both players, we call this game a Strictly Determined Game. The optimal strategy is for Player 1 to always play Row 3 and Player 2 to always play Row 4. The game has a payoff of -1 and is unfair to Player 2.

To Cheat or not to Cheat:
A Strictly Determined IRS Game

IRS Audits (columns) / **Taxpayer Cheats** (rows)

	Yes	No
Yes 1500	**-4500**	
No	**-200**	**0**

IRS Audits (columns) / **Taxpayer Cheats** (rows)

	Yes	No
Yes 1500	**-4500**	
No	**-200**	**0**

In this game, the taxpayer minimizes loses by always playing Row 1. Likewise, the IRS should always play Column 1. The value of the game is -$200.00 and is unfair to the taxpayer.

The Idea of Row/Column Dominance

1

-4	-2	1	2
-5	-2	-7	1
3	-1	0	3
1	-3	-1	5

2

-4	-2	1
-5	-2	-7
3	-1	0
1	-3	-1

3

-4	-2	1
3	-1	0

4

-4	-2
3	-1

5

3	-1

6

-1

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Non-Strictly Determined Game
Requiring a Mixed Strategy

	P2: Yes	P2: No
P1: Yes	**-50**	**40**
P1: No	**30**	**-20**

Scenario (assuming repeated play): Player 1 chooses a No which invites a No from Player 2. Player 1 responds with a switch to Yes, to which Player 2 answers with Yes, to which Player 1 answers with No, etc., etc., etc. The point is that each player can always defeat the other player’s next best play. This creates a loop of strikes and counterstrikes.

The Idea Behind Mixed Strategy

P2

	Col 1	Col 2
Row 1	-50	40
Row 2	30	-20

P1

Randomly mix the two choices--Row 1 or Row 2, Column 1 or Column 2-- such that the expected payoff remains constant (assuming repeated play)
over the long run, no matter what the other player does.
Basically, we are minimizing our surprises and adopting a Most Conservative Strategy.

What a Most Conservative Mixed Strategy Looks Like

P2

		q	1-q
		Col 1	Col 2
p	Row 1	-50	40
1-p	Row 2	30	-20

P1

Player 1 chooses p so that the payoff (expected value) remains constant no matter what Player 2 does. Likewise, Player 2 adopts the same strategy and chooses q to make it so.

So, how do we choose p and q?

Most Conservative Mixed Strategy: *Choosing p and q*

For the Row Player:

Payoff against Col 1 is -50p + 30(1-p)
Payoff against Col 2 is 40p - 20(1-p)

Now equate the two Payoffs

-50p + 30(1-p) = 40p - 20(1-p)

Solve for the p that makes it so.

p=5/14

Interpretation: Row Player should *randomly* play Row 1 and Row 2 per the Mixed Strategy (5/14, 9/14) in order to insure a long-term constant payoff.

For the Column Player:

Payoff against Row 1 is -50q + 40(1-q)
Payoff against Row 2 is 30q - 20(1-q)

Now equate the two Payoffs

-50q + 40(1-q) = 30q - 20(1-q)

Solve for the q that makes it so.

q=3/7

Interpretation: Column Player should *randomly* play Col 1 and Col 2 per the Mixed Strategy (3/7, 4/7) in order to insure a long-term constant payoff.

Most Conservative Mixed Strategy: *Minimax Theorem*

Row Player mixes per the strategy (p=5/14,1-p=9/14)

Payoff against Col 1 is -50p + 30(1-p)
= 20/14=1.4286.

Payoff against Col 2 is 40p - 20(1-p)
=20/14=1.4286

Column Player mixes per the strategy (q=3/7, 1-q=4/7)

Payoff against Row 1 is -50q + 40(1-q)
=10/7=1.4286

Payoff against Row 2 is 30q - 20(1-q)
=10/7=1.4286

Notice that the average payoff (expected value) identically reduces to 1.4286 when both players adopt a Most Conservative Mixed Strategy (*Minimax Theorem*). The game value is **1.4286**--unfair to column player!

Note: If one player deviates from the Most Conservative Mixed Strategy, the payoff still remains constant as long as the other player maintains a Most Conservative Mixed Strategy. However, the first player is now in a more vulnerable position.

Problem Utilizing all Elementary Game Theory Techniques

Stage 1: Using the idea of dominance to reduce the size of the game

$$\begin{bmatrix} -1 & -1 & 3 \\ 2 & 3 & -2 \\ -2 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 3 \\ 2 & -2 \\ -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$$

age 2: Finding the <u>Most Conservative Mixed Strategy</u>, value, and fairnes

$$\begin{array}{c c} & \begin{array}{cc} q & 1-q \end{array} \\ \begin{array}{c} p \\ 1-p \end{array} & \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \end{array}$$

Equate row expected values against remaining columns to obtain p=1/2

Equate column expected values against remaining rows to obtain q=5/8

<u>Most Conservative Mixed Strategy</u>

Row: (1/2, 1/2, 0)

Column: (5/8, 0, 3/8)

Value is 0.50

Unfair to Column Player